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Solutions of gauge invariant cosmological perturbations in long-wavelength limit

Yasusada Nambu* and Atsushi Taruya†

Department of Physics, Nagoya University

Chikusa, Nagoya 464-01, Japan

Abstract

We investigate gauge invariant cosmological perturbations in a spatially flat Friedman-Robertson-Walker universe with scalar fields. It is well known that the evolution equation for the gauge invariant quantities has exact solutions in the long-wavelength limit. We find that these gauge invariant solutions can be obtained by differentiating the background solution with respect to parameters contained in the background system. This method is very useful when we analyze the long-wavelength behavior of cosmological perturbation with multiple scalar fields.

*e-mail: nambu@allegro.phys.nagoya-u.ac.jp

†e-mail: ataruya@allegro.phys.nagoya-u.ac.jp

I. INTRODUCTION

The theory of gauge invariant cosmological perturbations [1–3] is an important tools when we investigate the origin and the evolution of the structure in our universe. In the context of general relativity, the treatment of linear perturbation of any matter field requires great care because these perturbations are in general not gauge invariant. We must evaluate gauge invariant quantities to extract physical meaning.

One of the main application of the gauge invariant perturbation is the theory of inflationary universe, in which the gauge invariant treatment of perturbations is crucial. The inflation predicts that the seed of density fluctuation, which evolves to the structures in our universe, is generated as quantum fluctuation of the inflaton field. The created fluctuations is stretched by deSitter expansion and their wavelength exceeds the Hubble horizon scale. As the gauge dependence of perturbations becomes conspicuous on super horizon scale, we must use gauge invariant perturbation to handle such a situation. But the derivation of the gauge invariant equation and evaluation of its solution needs tedious calculation in general, especially for the model with multiple scalar fields which is considered in the context of hybrid or extended inflation and scalar-tensor theories.

We here pay attention to the Einstein gravity with minimally coupled scalar fields ϕ. As a background space, we assume a flat Friedman-Robertson-Walker(FRW) universe of which dynamical variable is a scale factor $a = e^{\alpha}$. We use a following gauge invariant combination(Mukhanov's variable) [4]:

$$Q \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi, \tag{1}$$

where $\delta\phi$ is a perturbation of the scalar field, φ is a perturbation of three curvature and $H = \dot{\alpha}$ is a Hubble parameter. The evolution equation for Q is given by

$$\ddot{Q} + 3H\dot{Q} + \left[\left(\frac{k}{a}\right)^2 + V_{\phi\phi} - \frac{8\pi G}{e^{3\alpha}}\left(\frac{e^{3\alpha}}{H}\dot{\phi}^2\right)^{\cdot}\right] Q = 0. \tag{2}$$

It is known that this equation has exact solutions in the long-wavelength limit $k \to 0$ [5]:

$$Q = c_1 \frac{\dot{\phi}}{H} + c_2 \frac{\dot{\phi}}{H} \int^t \frac{H^2}{e^{3\alpha}\dot{\phi}^2} dt, \tag{3}$$

where c_1, c_2 are arbitrary constants. Therefore if we can obtain the background solution $(\alpha(t), \phi(t))$, it is possible to predict the behavior of the long-wavelength gauge invariant perturbation without solving the evolution equation of the perturbation.

Our question is why we can have the exact solution for $k \to 0$ (0-mode) gauge invariant perturbation. The 0-mode perturbation is a perturbation of homogeneous mode, it must be contained in a homogeneous background system. So we expect that the evolution equation for the 0-mode perturbation can be derived by the analysis of the mini-super space model which has no inhomogeneous mode.

In this paper, we aim at clarifying the relation between the gauge invariant 0-mode solution and the background solution. We will show that the evolution equation and the solution of the gauge invariant 0-mode perturbation can be obtained within the perturbation of mini-super space model. In Sec.II, we use one dimensional autonomous system to demonstrate that the perturbed equation and its solution is obtained by differentiating the background solution with respect to parameters contained in the background solution. In Sec.III, we treat a mini-super space model with scalar fields and derive the 0-mode gauge invariant perturbation equation(Mukhanov's equation) and its solutions. We use Hamilton-Jacobi method which is used by several authors [6,7] to derive the evolution equation of the gauge invariant cosmological perturbation. This method does not need to specify any gauge condition during calculation, it is suitable to apply to gauge invariant perturbation. Sec.IV is devoted to summary. We use the unit in which $c = \hbar = 8\pi G = 1$ and denote the partial derivative of a function F with respect to a some variable A by F_A throughout the paper.

II. TOY MODEL EXAMPLE

As a demonstration, we consider a one dimensional autonomous system with a Hamiltonian

$$\mathcal{H} = \frac{1}{2}p^2 + V(x). \tag{4}$$

Equation of motion is

$$\ddot{x} + V'(x) = 0. \tag{5}$$

The solution of this equation is written as $x = x(t + t_0)$ where t_0 is an origin of time and an arbitrary constant. The equation for the perturbation is obtained by splitting $x = x^{(0)} + \delta x$ and linearize with respect to δx:

$$\ddot{\delta x} + V''(x^{(0)})\delta x = 0. \tag{6}$$

This equation has the same form as the equation of $\dot{x}$ which is obtained by differentiating the background equation (5) with respect to time variable t: $(\dot{x})^{\cdot\cdot} + V''(x)\dot{x} = 0$. As $\dot{x} = x_{t_0}$, a solution of the perturbation equation is obtained by differentiating a background solution with respect to the parameter t_0.

We explain this result more rigorously using Hamilton-Jacobi(H-J) method which is suitable to treat a cosmological model in next section. The H-J equation for a generating function S is

$$\frac{\partial S}{\partial t} + \frac{1}{2}S_x^2 + V(x) = 0, \tag{7}$$

and the evolution equation is

$$\dot{x} = S_x. \tag{8}$$

Eq.(7) is solved by $S = -Et + W(x)$(E is a separation constant):

$$\begin{aligned} &W_x^2 + 2V(x) = 2E, \\ &\dot{x} = W_x. \end{aligned} \tag{9}$$

The solution is

$$t + t_0 = \int^x \frac{dx}{W_x} \equiv \tau(x; E), \tag{10}$$

where t_0 is an arbitrary constant. From this we get $x = x(t+t_0; E)$ or $E = E(x, t+t_0)$. The background solution is specified by the two parameters (t_0, E) and the perturbed solution is defined by the small change of the constants $(t_0 + \delta t_0, E + \delta E)$.

The generating function is written as $S = S^{(0)} + F$ where $S^{(0)}$ is a background part and F is a perturbed part which consists of quadratic terms of δt_0 and δE. F satisfies

$$\frac{\partial F}{\partial t} + S_x^{(0)} F_x + \frac{1}{2} F_x^2 = 0. \tag{11}$$

Now we change variables from (x, t) to $(\tau(x; E), t_0, E(x, t+t_0))$:

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial t_0}{\partial t}\frac{\partial}{\partial t_0} + \frac{\partial E}{\partial t}\frac{\partial}{\partial E} \\ &= -\frac{\partial}{\partial t_0} + E_t \frac{\partial}{\partial E},\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} + \frac{\partial t_0}{\partial x}\frac{\partial}{\partial t_0} + \frac{\partial E}{\partial x}\frac{\partial}{\partial E} \\ &= \tau_x \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial t_0}\right) + E_x \frac{\partial}{\partial E}.\end{aligned} \tag{13}$$

The Eq.(11) becomes

$$\frac{\partial F}{\partial \tau} + \frac{1}{2}\left(\tau_x\left(\frac{\partial F}{\partial \tau} + \frac{\partial F}{\partial t_0}\right) + E_x \frac{\partial F}{\partial E}\right)^2 = 0. \tag{14}$$

As we are interested in linear perturbation due to the change of the parameters $(\delta t_0, \delta E)$, we replace $\frac{\partial}{\partial t_0} \to \frac{\partial}{\partial \delta t_0}, \frac{\partial}{\partial E} \to \frac{\partial}{\partial \delta E}$. Using a new variable $\delta t \equiv \delta t_0 + \tau_E \delta E$ and taking into account the quadratic form of F, the H-J equation becomes

$$\frac{\partial F}{\partial \tau} + \frac{1}{2\dot{x}^2}\left(\frac{\partial F}{\partial \delta t}\right)^2 = 0, \tag{15}$$

From this equation, we get a Hamiltonian for the perturbation variable δt:

$$\mathcal{H}^{(2)} = \frac{1}{2\dot{x}^2} P_{\delta t}^2, \tag{16}$$

where $P_{\delta t}$ is a conjugate momentum of δt. The equation of motion is

$$\ddot{\delta t} + 2\frac{\ddot{x}}{\dot{x}}\dot{\delta t} = 0. \tag{17}$$

The equation for a variable $\delta x \equiv \dot{x}\delta t = x_{t_0}\delta t_0 + x_E \delta E$ becomes

$$\ddot{\delta x} + V''(x)\delta x = 0. \tag{18}$$

This is our desired result The two independent solutions are given by

$$\delta x = x_{t_0}, x_E. \tag{19}$$

The explicit form of the solution is obtained by differentiating Eq.(10) with respect t_0 and E. Regarding x as a function of $(t + t_0, E)$ and differentiating Eq.(10) with respect to t_0 and E,

$$\begin{aligned}
&\frac{\partial}{\partial t_0}: \quad 1 = x_{t_0}\frac{1}{W_x}, \\
&\frac{\partial}{\partial E}: \quad 0 = x_E\frac{1}{W_x} - \int^x \frac{W_{xE}}{W_x^2}dx,
\end{aligned} \tag{20}$$

and we have

$$\begin{aligned}
&x_{t_0} = W_x = \dot{x}, \\
&x_E = W_x\int^x \frac{dx}{W_x^3} = \dot{x}\int^t \frac{dt}{\dot{x}^2}.
\end{aligned} \tag{21}$$

The perturbed solution is obtained by differentiating the background solution with respect to the parameters which specify the background solution.

III. GAUGE INVARIANT LONG-WAVELENGTH SOLUTION FROM MINI-SUPER SPACE SOLUTION

A. a single scalar field case

We consider a spatially flat FRW model with a scalar field. Dynamical variables are a scale factor $a = e^\alpha$ and a scalar field ϕ. The Hamiltonian for this system is

$$\mathcal{H} = Ne^{-3\alpha}\left[-\frac{1}{12}P_\alpha^2 + \frac{1}{2}P_\phi^2 + e^{6\alpha}V(\phi)\right], \tag{22}$$

where N is a lapse function, P_α and P_ϕ are conjugate momentum of α and ϕ, $V(\phi)$ is an arbitrary potential for the scalar field. The equations of motion are

$$\begin{aligned}
\dot{\alpha} &= \frac{\partial \mathcal{H}}{\partial P_\alpha} = -\frac{N}{6} e^{-3\alpha} P_\alpha, \quad \dot{\phi} = \frac{\partial \mathcal{H}}{\partial P_\phi} = N e^{-3\alpha} P_\phi, \\
\dot{P}_\alpha &= -\frac{\partial \mathcal{H}}{\partial \alpha} = N e^{-3\alpha} \left[-\frac{1}{4} P_\alpha^2 + \frac{3}{2} P_\phi^2 - 3 e^{6\alpha} V(\phi) \right], \qquad (23) \\
\dot{P}_\phi &= -\frac{\partial \mathcal{H}}{\partial \phi} = N e^{3\alpha} V'(\phi).
\end{aligned}$$

The Hamilton-Jacobi equation follows from the Hamiltonian constraint with the canonical conjugate momentum replaced by derivative of a generating function S, $P_\alpha = \frac{\partial S}{\partial \alpha}, P_\phi = \frac{\partial S}{\partial \phi}$:

$$-\frac{S_\alpha^2}{12} + \frac{S_\phi^2}{2} + e^{6\alpha} V(\phi) = 0. \qquad (24)$$

The evolution equations are

$$\frac{\dot{\alpha}}{N} = -\frac{e^{-3\alpha}}{6} S_\alpha, \qquad (25)$$

$$\frac{\dot{\phi}}{N} = e^{-3\alpha} S_\phi. \qquad (26)$$

The background solution is obtained by assuming $S^{(0)} = -2 e^{3\alpha} H(\phi)$. Then

$$-3H^2 + 2H_\phi^2 + V(\phi) = 0, \qquad (27)$$

$$\frac{\dot{\alpha}}{N} = H, \qquad (28)$$

$$\frac{\dot{\phi}}{N} = -2H_\phi. \qquad (29)$$

The solution of Eq.(27) has a constant of integration C and the solution can be written $H = H(\phi; C)$. By differentiating Eq.(27) with respect to C and using the evolution equation (28) and (29), we have

$$H_C = D e^{-3\alpha} \equiv e^{-3(\alpha+\alpha_0)}, \qquad (30)$$

where D and α_0 are arbitrary constants. From this, we can express $\alpha + \alpha_0$ as a function of (ϕ, C):

$$\alpha + \alpha_0 \equiv \tilde{\alpha}(\phi; C). \qquad (31)$$

If we invert this relation, we get $\phi = \phi(\alpha + \alpha_0, C)$. α_0 is the origin of the scale factor and C determines the initial value of the scalar field ϕ. The background solution is characterized

by two parameters (α_0, C) and perturbation is defined by small change of these parameters $(\alpha_0+\delta\alpha_0, C+\delta C)$.

We split the generating function into the background and the perturbed part: $S = S^{(0)} + F$. F satisfies

$$-\frac{S^{(0)}_\alpha}{6}F_\alpha + S^{(0)}_\phi F_\phi - \frac{F_\alpha^2}{12} + \frac{F_\phi^2}{2} = 0. \tag{32}$$

Change variables from (α, ϕ) to $(\tilde{\alpha}, \alpha_0, C)$:

$$\begin{aligned}\frac{\partial}{\partial\alpha} &= \frac{\partial\tilde{\alpha}}{\partial\alpha}\frac{\partial}{\partial\tilde{\alpha}} + \frac{\partial\alpha_0}{\partial\alpha}\frac{\partial}{\partial\alpha_0} + \frac{\partial C}{\partial\alpha}\frac{\partial}{\partial C}\\ &= -\frac{\partial}{\partial\alpha_0} + C_\alpha\frac{\partial}{\partial C}, \qquad (33)\\ \frac{\partial}{\partial\phi} &= \frac{\partial\tilde{\alpha}}{\partial\phi}\frac{\partial}{\partial\tilde{\alpha}} + \frac{\partial\alpha_0}{\partial\phi}\frac{\partial}{\partial\alpha_0} + \frac{\partial C}{\partial\phi}\frac{\partial}{\partial C}\\ &= \tilde{\alpha}_\phi\left(\frac{\partial}{\partial\tilde{\alpha}} + \frac{\partial}{\partial\alpha_0}\right) + C_\phi\frac{\partial}{\partial C}. \qquad (34)\end{aligned}$$

Then the H-J equation (32) becomes

$$e^{3\alpha}H\frac{\partial F}{\partial\tilde{\alpha}} - \frac{1}{12}\left(-\frac{\partial F}{\partial\alpha_0} + C_\alpha\frac{\partial F}{\partial C}\right)^2 + \frac{1}{2}\left(\tilde{\alpha}_\phi\left(\frac{\partial F}{\partial\tilde{\alpha}} + \frac{\partial F}{\partial\alpha_0}\right) + C_\phi\frac{\partial F}{\partial C}\right)^2 = 0. \tag{35}$$

To extract the linear perturbation part, we replace $\frac{\partial}{\partial\alpha_0} \to \frac{\partial}{\partial\delta\alpha_0}, \frac{\partial}{\partial C} \to \frac{\partial}{\partial\delta C}$. Using a new variable $\mathcal{R} \equiv \delta\alpha_0 + \tilde{\alpha}_C\delta C$ and assuming a quadratic form of the generating function F with respect to $\mathcal{R}$, we have

$$\frac{\partial F}{\partial\tilde{\alpha}} + \frac{H}{8e^{3\alpha}H_\phi^2}\left(\frac{\partial F}{\partial\mathcal{R}}\right)^2 = 0. \tag{36}$$

From this, we get a Hamiltonian for the variable $\mathcal{R}$:

$$\mathcal{H}^{(2)} = \frac{H}{8e^{3\alpha}H_\phi^2}P_{\mathcal{R}}^2, \tag{37}$$

where $P_{\mathcal{R}}$ is a conjugate momentum of $\mathcal{R}$. The equation of the motion for $\mathcal{R}$ becomes

$$\mathcal{R}_{\alpha\alpha} + \left(3 + \frac{H_\alpha}{H} + 2\frac{\partial}{\partial\alpha}\ln\left(\frac{H_\phi}{H}\right)\right)\mathcal{R}_\alpha = 0. \tag{38}$$

By introducing a new variable $Q \equiv \phi_\alpha\mathcal{R} = \phi_{\alpha_0}\delta\alpha_0 + \phi_C\delta C$, Q satisfies

$$Q_{\alpha\alpha} + \left(3 + \frac{H_\alpha}{H}\right) Q_\alpha + \left(\frac{V_{\phi\phi}}{H^2} - \frac{1}{e^{3\alpha} H} \left(e^{3\alpha} H \phi_\alpha^2\right)_\alpha\right) Q = 0. \tag{39}$$

This is nothing but a Mukhanov equation for the long-wavelength limit. For 0-mode perturbation, the only gauge freedom is infinitesimal time coordinate transformation $t \to t + \delta t$. Then variables transform as $\delta\alpha \to \delta\alpha + \dot{\alpha}\delta t, \delta\phi \to \delta\phi + \dot{\phi}\delta t$. A possible gauge invariant combination of these variables is $\delta\alpha - (\dot{\alpha}/\dot{\phi})\delta\phi$ and this has a meaning of perturbation of 3 curvature in co-moving gauge (intrinsic curvature perturbation). In our calculation, we use $\tilde{\alpha}$ as a time parameter and this is equivalent to using co-moving gauge. Therefore the variable $\mathcal{R}$ is the gauge invariant intrinsic curvature perturbation or the Bardeen's parameter in the long-wavelength limit.

The two independent solutions are given by

$$Q = \phi_{\alpha_0}, \phi_C. \tag{40}$$

The explicit form of these solutions can be obtained using Eq.(27), (28), (29). Assuming that the lapse N is a function of the scalar field ϕ, we have

$$t + t_0 = -\frac{1}{2} \int^\phi \frac{d\phi}{N H_\phi}. \tag{41}$$

Regarding ϕ as a function of $(t + t_0, C)$ and differentiating the both side of this equation with respect to t_0 and C,

$$\begin{aligned} \frac{\partial}{\partial t_0} &: \quad 1 = -\frac{\phi_{t_0}}{2NH_\phi}, \\ \frac{\partial}{\partial C} &: \quad 0 = -\frac{\phi_C}{2NH_\phi} + \int^\phi \frac{H_{\phi C}}{2NH_\phi^2} d\phi. \end{aligned} \tag{42}$$

Choosing the scale factor as a time parameter($t \equiv \alpha, N \equiv 1/H$),

$$\begin{aligned} \phi_{\alpha_0} &= -2\frac{H_\phi}{H} = \phi_\alpha, \\ \phi_C &= NH_\phi \int^\phi \frac{H_{\phi C}}{NH_\phi^2} d\phi = -6\phi_\alpha \int^\alpha \frac{e^{-3(\alpha+\alpha_0)}}{\phi_\alpha^2 H} d\alpha. \end{aligned} \tag{43}$$

We therefore obtained the gauge invariant 0-mode solution by differentiating the background solution with respect to two parameters α_0 and C.

B. two scalar fields case

We consider a flat FRW universe with two scalar fields ϕ_1 and ϕ_2. The Hamilton-Jacobi equation is

$$-\frac{S_\alpha^2}{12}+\frac{S_{\phi_1}^2}{2}+\frac{S_{\phi_2}^2}{2}+e^{6\alpha}V(\phi_1,\phi_2)=0. \tag{44}$$

The background solution is obtained by $S^{(0)}=-2e^{3\alpha}H(\phi_1,\phi_2)$,

$$-3H^2+2H_{\phi_1}^2+2H_{\phi_2}^2+V(\phi_1,\phi_2)=0, \tag{45}$$

$$\frac{\dot{\alpha}}{N}=H, \tag{46}$$

$$\frac{\dot{\phi}_1}{N}=-2H_{\phi_1}, \quad \frac{\dot{\phi}_2}{N}=-2H_{\phi_2}. \tag{47}$$

The solution of Eq.(45) has two constants of integration C_1, C_2 and can be written $H=H(\phi_1,\phi_2,C_1,C_2)$. By differentiating the H-J equation (45) with respect to these constants and using the evolution equation (46) and (47), it can be shown

$$e^{3\alpha}H_{C1}=D_1, \quad e^{3\alpha}H_{C2}=D_2, \tag{48}$$

where D_1, D_2 are constants. Let us define new constants α_0 and f by $D_1=e^{-3\alpha_0}, D_2=e^{-3\alpha_0}f$. Then

$$H_{C1}=e^{-3(\alpha+\alpha_0)}, \tag{49}$$

$$\frac{H_{C2}}{H_{C1}}=f. \tag{50}$$

From this we have

$$\alpha+\alpha_0=\tilde{\alpha}(\phi_1,\phi_2,C_1,C_2), \tag{51}$$

$$f=f(\phi_1,\phi_2,C_1,C_2). \tag{52}$$

α_0 is the origin of scale factor. If C_1, C_2 are fixed, f determines trajectories in configuration space (ϕ_1,ϕ_2). We split the generating function into the background and the perturbed part: $S=S^{(0)}+F$. F satisfies

$$-\frac{S^{(0)}_{\alpha}}{6}F_{\alpha}+S^{(0)}_{\phi_1}F_{\phi_1}+S^{(0)}_{\phi_2}F_{\phi_2}-\frac{F^2_{\alpha}}{12}+\frac{F^2_{\phi_1}}{2}+\frac{F^2_{\phi_2}}{2}=0. \tag{53}$$

Change variables from (ϕ_1,ϕ_2,α) to $(\tilde{\alpha},\alpha_0,C_1,f,C_2)$:

$$\frac{\partial}{\partial\alpha}=-\frac{\partial}{\partial\alpha_0}+C_{1\alpha}\frac{\partial}{\partial C_1}+C_{2\alpha}\frac{\partial}{\partial C_2}, \tag{54}$$

$$\frac{\partial}{\partial\phi_1}=\tilde{\alpha}_{\phi_1}\left(\frac{\partial}{\partial\tilde{\alpha}}+\frac{\partial}{\partial\alpha_0}\right)+C_{1\phi_1}\frac{\partial}{\partial C_1}+f_{\phi_1}\frac{\partial}{\partial f}+C_{2\phi_1}\frac{\partial}{\partial C_2}, \tag{55}$$

$$\frac{\partial}{\partial\phi_2}=\tilde{\alpha}_{\phi_2}\left(\frac{\partial}{\partial\tilde{\alpha}}+\frac{\partial}{\partial\alpha_0}\right)+C_{1\phi_2}\frac{\partial}{\partial C_1}+f_{\phi_2}\frac{\partial}{\partial f}+C_{2\phi_2}\frac{\partial}{\partial C_2}. \tag{56}$$

The H-J equation becomes

$$\begin{aligned}&e^{3\alpha}H\frac{\partial F}{\partial\tilde{\alpha}}-\frac{1}{12}\left(-\frac{\partial F}{\partial\alpha_0}+C_{1\alpha}\frac{\partial F}{\partial C_1}+C_{2\alpha}\frac{\partial F}{\partial C_2}\right)^2\\&\quad+\frac{1}{2}\left(\tilde{\alpha}_{\phi_1}\left(\frac{\partial F}{\partial\tilde{\alpha}}+\frac{\partial F}{\partial\alpha_0}\right)+C_{1\phi_1}\frac{\partial F}{\partial C_1}+f_{\phi_1}\frac{\partial F}{\partial f}+C_{2\phi_1}\frac{\partial F}{\partial C_2}\right)^2\\&\quad+\frac{1}{2}\left(\tilde{\alpha}_{\phi_2}\left(\frac{\partial F}{\partial\tilde{\alpha}}+\frac{\partial F}{\partial\alpha_0}\right)+C_{1\phi_2}\frac{\partial F}{\partial C_1}+f_{\phi_2}\frac{\partial F}{\partial f}+C_{2\phi_2}\frac{\partial F}{\partial C_2}\right)^2=0\end{aligned} \tag{57}$$

To extract a linear perturbation part, we replace $\frac{\partial}{\partial\alpha_0}\to\frac{\partial}{\partial\delta\alpha_0}$, $\frac{\partial}{\partial f}\to\frac{\partial}{\partial\delta f}$, $\frac{\partial}{\partial C_1}\to\frac{\partial}{\partial\delta C_1}$, $\frac{\partial}{\partial C_2}\to\frac{\partial}{\partial\delta C_2}$. Using a new variable $\mathcal{R}\equiv\delta\alpha_0+\tilde{\alpha}_{C1}\delta C_1+\tilde{\alpha}_{C2}\delta C_2$ and assuming a quadratic form of F with respect to $\mathcal{R}$ and δf, we have

$$e^{3\alpha}H\frac{\partial F}{\partial\tilde{\alpha}}+\frac{1}{2}\left(\alpha_{\phi_1}\frac{\partial F}{\partial\mathcal{R}}+f_{\phi_1}\frac{\partial F}{\partial\delta f}\right)^2+\frac{1}{2}\left(\alpha_{\phi_2}\frac{\partial F}{\partial\mathcal{R}}+f_{\phi_2}\frac{\partial F}{\partial\delta f}\right)^2=0. \tag{58}$$

Define 2×2 matrices

$$\hat{X}=\begin{bmatrix}\phi_{1\alpha}, & \phi_{1f}\\ \phi_{2\alpha}, & \phi_{2f}\end{bmatrix},\quad \hat{A}=\frac{1}{e^{3\alpha}H}\left(\hat{X}^{-1}\right)\left(\hat{X}^{-1}\right)^T. \tag{59}$$

The H-J equation becomes

$$\frac{\partial F}{\partial\tilde{\alpha}}+\frac{1}{2}\begin{bmatrix}\frac{\partial F}{\partial\mathcal{R}} & \frac{\partial F}{\partial\delta f}\end{bmatrix}\hat{A}\begin{bmatrix}\frac{\partial F}{\partial\mathcal{R}}\\ \frac{\partial F}{\partial\delta f}\end{bmatrix}=0. \tag{60}$$

We get a Hamiltonian for the perturbation variables $(\mathcal{R},\delta f)$:

$$\mathcal{H}^{(2)}=\frac{1}{2}\begin{bmatrix}P_{\mathcal{R}} & P_{\delta f}\end{bmatrix}\hat{A}\begin{bmatrix}P_{\mathcal{R}}\\ P_{\delta f}\end{bmatrix}, \tag{61}$$

where $P_{\mathcal{R}}$ and $P_{\delta f}$ are conjugate momentum of $\mathcal{R}$ and δf, respectively. The evolution equation becomes

$$\begin{bmatrix} \mathcal{R} \\ \delta f \end{bmatrix}_{\alpha\alpha} + \hat{A}\left(\hat{A}^{-1}\right)_{\alpha} \begin{bmatrix} \mathcal{R} \\ \delta f \end{bmatrix}_{\alpha} = 0, \tag{62}$$

Introducing a new variable

$$\vec{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \equiv \hat{X} \begin{bmatrix} \mathcal{R} \\ \delta f \end{bmatrix}, \tag{63}$$

$\vec{Q}$ satisfies

$$\vec{Q}_{\alpha\alpha} + \left(3 + \frac{H_\alpha}{H}\right)\vec{Q}_\alpha + \hat{M}\vec{Q} = 0, \tag{64}$$

where $\hat{M}$ is a 2×2 matrix defined by

$$\left(\hat{M}\right)_{ij} = \frac{V_{\phi_i\phi_j}}{H^2} - \frac{1}{e^{3\alpha}H}\left(e^{3\alpha}H\phi_{i\alpha}\phi_{j\alpha}\right)_{\alpha}. \tag{65}$$

This is the Mukhanov equation for two scalar field system in the long-wavelength limit [8]. The four independent solutions of this equation are

$$\vec{Q} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{\alpha_0}, \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{f}, \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{C_1}, \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{C_2}. \tag{66}$$

These solutions can be written in another form:

$$\vec{Q} = \hat{X}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \hat{X}\int^{\alpha} \frac{1}{e^{3\alpha}H}\left(\hat{X}^{-1}\right)\left(\hat{X}^{-1}\right)^{T} d\alpha \begin{bmatrix} d_3 \\ d_4 \end{bmatrix}, \tag{67}$$

where d_1, d_2, d_3, d_4 are arbitrary constants. Using the Mukhanov's variable, $\mathcal{R}$ and δf can be expressed as

$$\begin{bmatrix} \mathcal{R} \\ \delta f \end{bmatrix} = \hat{X}^{-1}Q = \begin{bmatrix} \alpha_{\phi_1}Q_1 + \alpha_{\phi_2}Q_2 \\ f_{\phi_1}Q_1 + f_{\phi_2}Q_2 \end{bmatrix}. \tag{68}$$

In two scalar field case, using $\tilde{\alpha}$ as a time parameter means

$$\delta\tilde{\alpha} = \tilde{\alpha}_{\phi_1}\delta\phi_1 + \tilde{\alpha}_{\phi_2}\delta\phi_2 = 0. \tag{69}$$

From the relation

$$\tilde{\alpha}_{\phi_i} = \frac{\partial\tilde{\alpha}}{\partial H}\frac{\partial H}{\partial\phi_i} = -\frac{H}{2H_\alpha}\phi_{i,\alpha}, \tag{70}$$

we have $\delta\tilde{\alpha} = \sum_i \dot{\phi}_i\delta\phi_i = 0$. So using $\tilde{\alpha}$ as a time parameter is equivalent to using co-moving gauge and the variable $\mathcal{R}$ is the gauge invariant intrinsic curvature perturbation in the long-wavelength limit.

We can rewrite the expression of $\mathcal{R}$ using Eq.(70):

$$\mathcal{R} = -\frac{H}{2H_\alpha}(\phi_{1\alpha}Q_1 + \phi_{2\alpha}Q_2). \tag{71}$$

This gives a relation between the curvature perturbation $\mathcal{R}$ which corresponds to the Bardeen's parameter ζ and the Mukhanov's variable Q_1, Q_2 in the long-wavelength limit. Q_1, Q_2 are given by Eq.(67) which is written by using background solution. The important point is that we have not assumed any approximation to derive the long-wavelength solution. We can get the gauge invariant gravitational potential Φ by considering perturbation of the evolution equation (46):

$$\delta\dot{\alpha} - H\delta N = \sum_i H_{\phi_i}\delta\phi_i. \tag{72}$$

In longitudinal gauge, $\delta N = -\delta\alpha \equiv \Phi$ and $Q_i = \delta\phi_i + (\dot{\phi}_i/H)\Phi$. The above equation becomes

$$\dot{\Phi} + H\Phi = \frac{1}{2}\sum_i \dot{\phi}_i\delta\phi_i. \tag{73}$$

The curvature perturbation $\mathcal{R}$ can be written using Φ:

$$\begin{aligned}\mathcal{R} &= -\frac{H}{2\dot{H}}\sum_i \dot{\phi}_i Q_i \\ &= -\frac{H}{2\dot{H}}\sum_i\left(\dot{\phi}_i\delta\phi_i + \frac{\dot{\phi}_i^2}{H}\Phi\right) \\ &= \Phi - \frac{H}{\dot{H}}(\dot{\Phi} + H\Phi). \end{aligned}\tag{74}$$

Therefore we have

$$\Phi = -\frac{H}{e^\alpha}\int dt e^\alpha \frac{\dot{H}}{H^2}\mathcal{R} \tag{75}$$

C. application to inflationary model

We demonstrate our 0-mode solution (66) of perturbation reproduces previously obtained results [10–12] in inflationary model using the slow rolling approximation. If we concentrate on growing mode solutions, the gauge invariant 0-mode solution is given by

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = A \begin{bmatrix} \phi_{1\alpha} \\ \phi_{2\alpha} \end{bmatrix} + B \begin{bmatrix} \phi_{1f} \\ \phi_{2f} \end{bmatrix}, \tag{76}$$

where A, B are arbitrary constants. Substituting this solution to the expression of the curvature perturbation (71), we have

$$\mathcal{R} = A - B\frac{H}{2H_\alpha}\left(\phi_{1\alpha}\phi_{1f} + \phi_{2\alpha}\phi_{2f}\right). \tag{77}$$

The first term is the contribution of adiabatic fluctuations and remains constant in time. The second term is the contribution due to iso-curvature or entropic fluctuations which arises whenever there are two or more scalar fields. Using the slow rolling approximation, H-J equation and evolution equations become

$$\begin{aligned} H^2 &\approx \frac{1}{3}V(\phi_1,\phi_2), \\ \phi_{1\alpha} &= -\frac{2}{H}H_{\phi 1} = -\frac{V_{\phi 1}}{V}, \\ \phi_{2\alpha} &= -\frac{2}{H}H_{\phi 2} = -\frac{V_{\phi 2}}{V}. \end{aligned} \tag{78}$$

In this approximation, we do not have parameters C_1, C_2 contained in the Hubble function H and we neglect decaying mode solutions of perturbations. In the inflationary phase, $\dot{H}/H \ll 1$ and the gauge invariant gravitational potential becomes

$$\Phi \approx \frac{\dot{H}}{H^2}\mathcal{R} \approx -\frac{V_{\phi 1}^2 + V_{\phi 2}^2}{2V^2}\mathcal{R}. \tag{79}$$

For general form of the potential $V(\phi_1,\phi_2)$, we can not write down the solution of the evolution equation (78). We consider two examples of the potential in which case we have integral form of the slow rolling solution.

<u>Case 1</u>: $V(\phi_1,\phi_2) = V_1(\phi_1) + V_2(\phi_2)$. By integrating the evolution equation (78), we have

$$\alpha + \alpha_0 = -\int d\phi_1 \frac{V_1}{V_{1,\phi 1}} - \int d\phi_2 \frac{V_2}{V_{2,\phi 2}}, \tag{80}$$

$$f = \int \frac{d\phi_1}{V_{1,\phi 1}} - \int \frac{d\phi_2}{V_{2,\phi 2}}, \tag{81}$$

where α_0, f are constants of integration. By differentiating the above solution with respect to the parameter f, we have

$$\begin{aligned} \phi_{1f} &= \frac{V_2}{V_1 + V_2} V_{1,\phi 1}, \\ \phi_{2f} &= -\frac{V_1}{V_1 + V_2} V_{2,\phi 2}. \end{aligned} \tag{82}$$

Substituting these solutions to the expression of $\mathcal{R}$ (77), we have

$$\mathcal{R} = A - B \frac{\left(V_{1,\phi 1}\right)^2 V_2 - \left(V_{2,\phi 2}\right)^2 V_1}{\left(V_{1,\phi 1}\right)^2 + \left(V_{2,\phi 2}\right)^2}. \tag{83}$$

The gauge invariant gravitational potential is

$$\Phi = A \frac{H_\alpha}{H} - \frac{B}{2} \frac{V_{1,\alpha} V_2 - V_{2,\alpha} V_1}{V_1 + V_2}. \tag{84}$$

Case 2: $V(\phi_1, \phi_2) = V_1(\phi_1) V_2(\phi_2)$. By integrating the evolution equation (78), we have

$$\begin{aligned} \alpha + \alpha_0 &= -\int d\phi_1 \frac{V_1}{2V_{1,\phi 1}} - \int d\phi_2 \frac{V_2}{2V_{2,\phi 2}}, \\ f &= \int d\phi_1 \frac{V_1}{V_{1,\phi 1}} - \int d\phi_2 \frac{V_2}{V_{2,\phi 2}}, \end{aligned} \tag{85}$$

where α_0, f are constants of integration. By differentiating the above equations with respect to f, we have

$$\begin{aligned} \phi_{1,f} &= \frac{V_{1,\phi 1}}{2V_1}, \\ \phi_{2,f} &= -\frac{V_{2,\phi 2}}{2V_2}. \end{aligned} \tag{86}$$

Substituting these solutions to the expression of $\mathcal{R}$ (77), we have

$$\mathcal{R} = A - \frac{B}{2} \frac{\left(\frac{V_{1,\phi 1}}{V_1}\right)^2 - \left(\frac{V_{2,\phi 2}}{V_2}\right)^2}{\left(\frac{V_{1,\phi 1}}{V_1}\right)^2 + \left(\frac{V_{2,\phi 2}}{V_2}\right)^2}. \tag{87}$$

The gauge invariant gravitational potential is

$$\Phi = -\frac{1}{2}\left(A + \frac{B}{2}\right)\left(\frac{V_{1,\phi 1}}{V_1}\right)^2 - \frac{1}{2}\left(A - \frac{B}{2}\right)\left(\frac{V_{2,\phi 2}}{V_2}\right)^2. \tag{88}$$

IV. SUMMARY

We have shown that the solution and the evolution equation of the gauge invariant perturbation in the long-wavelength limit can be derived within the mini-super space model. The solution is obtained by differentiating the background solution with respect to parameters contained in the background system. These parameters completely specify the background solutions. If we need to know the behavior of the long-wavelength perturbation only, this method is very useful because we does not have to solve the perturbation equation. What we have to do is to solve the background system and identify the parameters contained in the background solution. The analysis of the long-wavelength perturbation in the model that involves multiple scalar field becomes easier.

The most important feature of our 0-mode solution is it does not rely on the slow roll approximation which is usually assumed to analyze the perturbation in the multi-field inflationary model [10–13]. As an application, a model of reheating after inflation was considered [9]. The system consists of a oscillating inflaton field and a massless scalar field which interacts with the inflaton. By using the approximation which can include non-linear effect, we obtained background solution and the behavior of the long-wavelength curvature perturbation was obtained using the method presented in this paper.

As a straightforward extension, it is possible to treat spatially closed and open FRW universe and anisotropic Kasner type universe. Instead of scalar fields, perfect fluid can be included. These subjects are left as a future exercise.

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